Spring 2017 MATH5012

Real Analysis II

Solution to Exercise 3

(1) For $a, b > 0$, set

$$f(x) = \begin{cases} x^a \sin(x^{-b}), & 0 < x \leq 1 \\ 0, & x = 0. \end{cases}$$

Show that f is in $BV[0,1]$ iff $a > b$.

Solution. Put $x_n = \left(\frac{1}{n\pi+\pi/2}\right)^{\frac{1}{b}}$ for $n \geq 0$. We claim that f is of bounded variation on $[0, x_0]$, hence of bounded variation on $[0,1]$. Now, note that $|\sin x_n| = 1$ and $\sin x_n \sin x_{n-1} = -1$, the total variation of f over $[0, x_0]$ is

$$\sum_{n=1}^{\infty} |f(x_{n-1}) - f(x_n)| = \sum_{n=1}^{\infty} \left| \left(\frac{1}{(n-1)\pi + \pi/2}\right)^{\frac{a}{b}} + \left(\frac{1}{n\pi + \pi/2}\right)^{\frac{a}{b}} \right|$$

We have

$$\sum_{n=1}^{\infty} \left| \left(\frac{1}{n\pi + \pi/2}\right)^{\frac{a}{b}} \right| \leq \sum_{n=1}^{\infty} |f(x_{n-1}) - f(x_n)| \leq 2\sum_{n=1}^{\infty} \left| \left(\frac{1}{(n-1)\pi + \pi/2}\right)^{\frac{a}{b}} \right|$$

It is easy to verify that the partial sum of the series converges iff $a > b$ hence the total variation is bounded iff $a > b$. We in fact can show that $f \in AC[0,1]$ if $a > b$, since for $x \neq 0$

$$f'(x) = ax^{a-1}\sin(x^{-b}) - bx^{a-b-1}\cos(x^{-b}) \in \mathcal{L}^1[0,1].$$

Now let $x \in (0,1]$ and $0 < \varepsilon < x$, by the fundamental theorem of calculus for smooth function, we have

$$f(x) - f(\varepsilon) = \int_{\varepsilon}^{x} f'(t) d\mathcal{L}^1(t) \ .$$

By the continuity of f at 0, we may let $\varepsilon \to 0$. Hence

$$f(x) - f(0) = \int_0^x f' d\mathcal{L}^1$$

and the absolute continuity of f follows from Theorem 6.18.

(2) A function is called Lipschitz continuous on an interval I if $\exists M > 0$ such that

$$|f(x) - f(y)| \leq M |x - y| .$$

(a) Show that every Lipschitz continuous function is absolutely continuous on I.

(b) Show that there are always absolutely continuous functions which are not Lipschitz continuous.

Solution.

(a) It follows directly from

$$\sum |f(x_i) - f(x_i')| \leq M \sum |x_i - x_i'|.$$

(b) Put $f(x) = \sqrt{x}$ on $[0, 1]$. Then we have

$$f(x) = \int_0^x \frac{1}{2\sqrt{t}} dt$$

is an indefinite integral hence is absolutely continuous. But that

$$\frac{f(x) - f(0)}{x - 0} = \frac{1}{\sqrt{x}} \to \infty \text{ as } x \to 0$$

shows that f is not Lipschitz.

(3) Assume that $1 < p < \infty$, f is absolutely continuous on $[a, b]$, $f' \in L^p$, and $\alpha = 1/q$, where q is the exponent conjugate to p. Prove that $f \in \text{Lip}\alpha$.

Solution. Since f is absolutely continuous, we have

$$\begin{aligned} |f(y)-f(x)| = \left|\int_x^y f'\right| &\leq \int_x^y |f'| \\ &\leq \left(\int_x^y |f'|^p\right)^{\frac{1}{p}} \left(\int_x^y 1\right)^{\frac{1}{q}} \\ &\leq |y-x|^{\frac{1}{q}} \|f'\|_p \end{aligned}$$

by Hölder's inequality.

Note. Nowadays the standard notation is $f \in C^\alpha[a,b],\ \alpha = 1/q$. This is a simple version of embedding inequality.

(4) Show that the product of two absolutely continuous functions on $[a,b]$ is absolutely continuous. Use this to derive a theorem about integration by parts.

Solution. Let f, g be absolutely continuous. Since every absolutely continuous function is bounded on every bounded closed interval, we can suppose that $|f|$, $|g| \leq M$.

$$\begin{aligned} \sum |f(x_i)g(x_i) - f(x_i')g(x_i')| &\leq \sum |f(x_i)g(x_i) - f(x_i)g(x_i')| + \sum |f(x_i)g(x_i') - f(x_i')g(x_i')| \\ &\leq M \sum |g(x_i) - g(x_i')| + M \sum |f(x_i) - f(x_i')|. \end{aligned}$$

This shows that fg is absolutely continuous whenever f, g are absolutely continuous. Integrating

$$(fg)' = f'g + fg'$$

(this is the product rule for differentiable functions, still valid) from a to b gives

$$f(b)g(b) - f(a)g(a) - \int_a^b fg' = \int_a^b f'g$$

since fg is absolutely continuous.

(5) Suppose $E \subset [a,b]$, $m(E) = 0$. Construct an absolutely continuous monotonic

function f on $[a,b]$ so that $f'(x) = \infty$ at every $x \in E$.

Hint: $E \subset \bigcap V_n$, V_n open, $m(V_n) \leq 2^{-n}$. Consider the sum of the characteristic functions of these sets.

Solution. Since $\mathcal{L}(E) = 0$, there exists a decreasing sequence $\{V_n\}$ of open set with $\mathcal{L}(V_n) < 2^{-n}$ such that

$$E \subseteq \bigcap V_n.$$

Put

$$f(x) = \int_a^x \sum_n \chi_{V_n}.$$

let $x \in E$. Choosing decreasing $\delta_n > 0$ such that $W_n = (x - \delta_n, x + \delta_n) \subseteq V_n$. If $0 < h < \delta_N$,

$$f(x+h) - f(x) = \int_x^{x+h} \sum_{n=1}^{\infty} \chi_{V_n} \geq \int_x^{x+h} \sum_{n=1}^{N} \chi_{W_n} \geq Mh \ ,$$

where M is the number of W_n containing $(x-h, x+h)$. Clearly $M \to \infty$ as $h \to 0$.

The same holds for $\delta_N < h < 0$. This shows that $\frac{f(x+h)-f(x)}{h} \to \infty$ at every $x \in E$.

(6) Let f be in $AC[a,b]$. Show that the total variation for f of f is also in $AC[a,b]$. Moreover,

$$T_f(b) = \int_a^b |f'(t)| \ dt.$$

Solution. The inequality $T_f(b) \leq \int_{[a,b]} |f'|$ is straightforward, so we work on the other inequality.

We first assume f' is continuous. For $\varepsilon > 0$,, let $A = \{x : \ f'(x) > \varepsilon\}, \ B = \{x :$

$f'(x) < -\varepsilon\}$, and $C = \{x : |f'(x)| \le \varepsilon\}$. We have disjoint decompositions

$$A = \dot{\bigcup}_j (a_j, b_j), \quad B = \dot{\bigcup}_j (c_j, d_j) \ .$$

Then

$$\begin{aligned} \int_a^b |f'| &= \int_A f' + \int_B (-f') + \int_C |f'| \\ &= \sum_j \int_{a_j}^{b_j} f' + \sum_j \int_{c_j}^{d_j} (-f') + \int_C |f'| \\ &\le \sum_j |f(b_j) - f(a_j)| + \sum_j |f(d_j) - f(c_j)| + \varepsilon(b-a) \\ &\le T_f(b) + \varepsilon(b-a) \ , \end{aligned}$$

which implies

$$T_f(b) \ge \int_a^b |f'| \ .$$

In general, pick a continuous φ so that $\|f' - \varphi\|_{L^1} < \varepsilon$ and define $g(x) = \int_a^x \varphi$. Then $|f(x) - g(x)| \le C\varepsilon$ on $[a, b]$ for some constant C. Using $|T_f(b) - T_g(b)| \le T_{f-g}(b) \le C'\varepsilon$ one can show that $T_f(b) \ge \int_a^b |f'|$ in the general case.

(7) Let X and Y be topological spaces having countable bases.

(a) Show that $X \times Y$ has a countable base. (In product topology on $X \times Y$, a set G is open if $\forall (x, y) \in G$, there is some G_1 open in X, G_2 open in Y such that $(x, y) \in G_1 \times G_2 \subset G$.)

(b) Let μ and ν be Borel measures on X and Y respectively. Show that $\mu \times \nu$ is a Borel measure.

Solution.

(a) Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be the countable bases of X and Y respectively. Put $\mathcal{B}$ be the collection of all $U \times V$ where $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$. We know that $\mathcal{B}$ is countable. We claim that $\mathcal{B}$ forms a base for $X \times Y$. Let W be open in

$X \times Y$. Let $(x, y) \in W$. Then there exists open U, V containing x and y respectively. Choose $U_1 \in \mathcal{B}_X$ be a subset of U and $V_1 \in \mathcal{B}_Y$ be a subset of V. Then $U_1 \times V_1 \in \mathcal{B}$ and $(x, y) \in U_1 \times V_1 \subseteq W$.

(b) As in the proof in (a), every open set in $X \times Y$ is a countable union of members in $\mathcal{B}$. But each member in $\mathcal{B}$ is a measurable rectangle because μ and ν are both Borel. It follows that every open set hence every Borel set is measurable.

(8) Let μ be the product measure $\mathcal{L}^1 \times \cdots \times \mathcal{L}^1$ on $\mathbb{R}^n$. Show that μ is equal to $\mathcal{L}^n$.

Solution. Both measures satisfy the following characterization of the Lebesgue measure: (a) translational invariant and (b) the measure of the unit square is 1. See an exercise in last semester.

Note. The definition of the product measure of finitely many measures can be done like the $n = 2$ case. One can also show that $\mathcal{L}^n \times \mathcal{L}^m = \mathcal{L}^{n+m}$.

(9) Fix $a_1 = 0 < a_2 < a_3 < \cdots < a_n \uparrow 1$ and let g_n be a continuous function, $\text{spt} g_n \subset (a_n, a_{n+1})$, $n \geq 1$, $\displaystyle\int g_n = 1$. Let

$$f(x, y) = \sum_{n=1}^{\infty} (g_n(x) - g_{n+1}(x)) g_n(y).$$

Verify that

$$\int \left(\int f \, dx \right) dy = 0, \text{ but}$$
$$\int \left(\int f \, dy \right) dx = 1,$$

and f is $\mathcal{L}^2$-measurable. Explain why Fubini's theorem cannot apply.

Solution. Since for each (x, y), only the N-th term of the sum is non-zero if

$y \in (a_N, a_{N+1})$. Now for each y lies in support of g_N,

$$\int f(x,y)dx = \int_{a_N}^{a_{N+1}} [g_N(x) - g_{N+1}(x)]g_N(y)dx = 0$$

because $\int g_n = 1$ for all n. Thus

$$\int\int f(x,y)dxdy = 0.$$

If y does not lie in any support of g_n, the same conclusion holds. For x lies in support of g_N, if $N > 1$

$$\int f(x,y)dy = \int -g_N(x)g_{N-1}(y) + g_N(x)g_N(y)dy = 0$$

and if $N = 1$,

$$\int f(x,y)dy = \int_{a_1}^{a_2} g_1(x)g_1(y)dy = g_1(x).$$

If follows that

$$\int\int f(x,y)dydx = \int_{a_1}^{a_2}\int f(x,y)dydx = 1.$$

That f is $\mathcal{L}^2$-measurable follows from the fact that f is a countable sum of continuous functions. Fubini's fails since

$$\begin{aligned}\int |f(x,y)|dy &= \int |-g_N(x)g_{N-1}(y) + g_N(x)g_N(y)|dy \\ &= \int_{a_{N-1}}^{a_N} |g_N(x)g_{N-1}(y)|dy + \int_{a_N}^{a_{N+1}} |g_N(x)g_N(y)|dy \\ &\geq 2|g_N(x)|\ ,\end{aligned}$$

if $x \in (a_N, a_{N+1})$, which shows that

$$\int\int |f(x,y)|dydx = \sum_{n=1}^{\infty}\int_{a_n}^{a_{n+1}} 2|g_n(x)|dx = \infty.$$

(10) Let μ and ν be outer measures defined on X and Y respectively and let f be μ-measurable and g ν-measurable with values in $(-\infty, \infty]$. Is it true that $(x, y) \mapsto f(x)+g(y)$ measurable in $\mu\times\nu$? How about the map $(x, y) \mapsto f(x)g(y)$?

Solution. Yes, they are. Since the sum and product of $\mu \times \nu$ measurable functions are $\mu\times\nu$ measurable. It suffices to check the functions $F(x, y) := f(x)$ and $G(x, y) := g(y)$ are $\mu \times \nu$ measurable. W.L.O.G. we only consider F. Let U be an open set, then

$$F^{-1}(U) = f^{-1}(U) \times Y \text{ is } \mu \times \nu \text{ measurable}$$

since it is a measurable rectangle. Hence we have F is $\mu\times\nu$ measurable function.

(11) (a) Suppose that f is a real-valued function in $\mathbb{R}^2$ such that each section f_x is Borel measurable and each section f^y is continuous. Prove that f is Borel measurable in $\mathbb{R}^2$. There is a hint given in [R1].

(b) Suppose that g is a real-valued function in $\mathbb{R}^n$ which is continuous in each of the n-variables separately. Prove that g is Borel.

Solution.

(a) We try to show that f is the pointwise limit of a sequence of Borel measurable functions and so is Borel measurable. We define f_n piecewisely in the following way: if $a_{i-1} := \frac{i-1}{n} \le x < \frac{i}{n}$, for some $i \in \mathbb{Z}$, then

$$f_n(x, y) = \frac{a_i - x}{a_i - a_{i-1}} f(a_{i-1}, y) + \frac{x - a_{i-1}}{a_i - a_{i-1}} f(a_i, y).$$

Obviously f_n is Borel measurable by the previous problem and satisfies

$$|f_n(x, y) - f(x, y)| \le |f(a_{i-1}, y) - f(x, y)| + |f(a_i, y) - f(x, y)| \to 0 \text{ as } n \to \infty$$

since we have f^y is a continuous function.

(b) We prove the result by doing induction on n. The case for $n = 1$ is trivial. Suppose it is true for $n = k$. For $n = k + 1$, the map $y := (x_2, \cdots x_{k+1}) \mapsto g(x_1, \cdots, x_{k+1})$ is Borel measurable by induction assmption, i.e. g_{x_1} is Borel measurable. Moreover, by our assumption g^y is continuous. g is Borel follows directly by considering the following sequence g_m as before,

$$g_m(x_1, y) = \frac{a_i - x_1}{a_i - a_{i-1}} g(a_{i-1}, y) + \frac{x_1 - a_{i-1}}{a_i - a_{i-1}} g(a_i, y).$$

(12) Suppose that f is real-valued in $\mathbb{R}^2$, f_x is Lebesgue measurable for each x, and f^y is continuous for each y. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, and put $h(y) = f(g(y), y)$. Prove that h is Lebesgue measurable on $\mathbb{R}$.

Solution. We argue as before to define h_n by

$$h_n(y) := f_n(g(y), y) = \frac{a_i - g(y)}{a_i - a_{i-1}} f(a_{i-1}, y) + \frac{g(y) - a_{i-1}}{a_i - a_{i-1}} f(a_i, y)$$

if $g(y) \in [a_{i-1}, a_i)$. Obviously h_n is Lebesgue measurable and tends to h pointwisely as n goes to ∞. Therefore, $h(y)$ is Lebesgue measurable.

(13) Give an example of two measurable sets A and B in $\mathbb{R}^2$ but $A + B$ is not measurable.

Suggestion: For the two-dimensional case, take $A = \{0\} \times [0, 1]$ and $B = \mathbb{N} \times \{0\}$ where $\mathbb{N}$ is a non-measurable set in $\mathbb{R}$.

Solution. Let $\mathbb{N}$ be the non-measurable set in $\mathbb{R}$ constructed by picking exactly one element in $[0, 1]$ from each equivalence class defined by

$$x \sim y \text{ iff } x - y \in \mathbb{Q}.$$

Put $A = \{0\} \times [0, 1]$ and $B = \mathbb{N} \times \{0\}$. Then since A and B are of measure zero, they are both measurable. But

$$A + B = \mathbb{N} \times [0,1].$$

We write $\{r_n\}$ is an enumeration of $\mathbb{Q}$ in $[-1,1]$. We have

$$[0,1] \times [0,1] \subseteq \bigcup (A + B + (r_i, 0)) \subseteq [-1,2] \times [0,1]$$

which shows that it is impossible to have countable additivity for the union above.